

Characterizations of special skew ruled surfaces by the normal curvature of some distinguished families of curves

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Abstract

We consider skew ruled surfaces in the three-dimensional Euclidean space and some geometrically distinguished families of curves on them whose normal curvature has a concrete form. The aim of this paper is to find and classify all ruled surfaces with the mentioned property.

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1 Introduction

Geometrically distinguished families of curves on a skew ruled surface in the Euclidean space \mathbb{R}^3 have been studied by a number of authors and in many points of view. A range of results appears when one requires that the curves of the considered family possess an additional property. The present paper contributes to this field of themes. We consider special families of curves on a skew ruled surface and suppose that the normal curvature along these curves has a concrete form. Our aim is to find the type of all ruled surfaces with the mentioned property and to classify them. The results are assembled in the table at the end of the paper.

To set the stage for this work the classical notation of ruled surface theory is briefly presented; for this purpose [3] is used as a general reference. In the Euclidean space \mathbb{R}^3 let Φ be a regular ruled surface without torsal rulings determined on $G := I \times \mathbb{R}$ ($I \subset \mathbb{R}$ open interval) and of the class C^3 . Φ can be expressed in terms of the striction line $\mathbf{s} = \mathbf{s}(u)$ and the unit vector field $\mathbf{e}(u)$ pointing along the rulings as

$$\mathbf{x}(u, v) = \mathbf{s}(u) + v \mathbf{e}(u), \quad u \in I, \quad v \in \mathbb{R}. \quad (1)$$

Moreover we can choose the parameter u to be the arclength along the spherical curve $e(u)$. Putting $f'(u) = \frac{df}{du}$ for a differentiable function $f(u)$ we have

$$\langle s'(u), e'(u) \rangle = 0, \quad |e'(u)| = 1 \quad \forall u \in I, \quad (2)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in \mathbb{R}^3 . The *conical curvature* $k(u)$, the *parameter of distribution* $\delta(u)$ and the *striction* $\sigma(u)$ of the surface Φ are given by

$$k(u) = (e(u), e'(u), e''(u)), \quad \delta(u) = (e(u), e'(u), s'(u)), \quad \sigma(u) := \angle(e(u), s'(u)),$$

where

$$-\frac{\pi}{2} < \sigma \leq \frac{\pi}{2} \quad \text{and} \quad \text{sign } \sigma = \text{sign } \delta.$$

The functions $k(u)$, $\delta(u)$ and $\sigma(u)$ consist a complete system of invariants of the surface Φ ([3]; p.19).

The components g_{ij} and h_{ij} of the first and the second fundamental tensors in (local) coordinates $u^1 := u$, $u^2 := v$ are the following

$$\begin{cases} (g_{ij}) = \begin{pmatrix} v^2 + \delta^2(\lambda^2 + 1) & \delta\lambda \\ \delta\lambda & 1 \end{pmatrix} \\ (h_{ij}) = \frac{1}{w} \begin{pmatrix} -[k v^2 + \delta'v + \delta^2(k - \lambda)] & \delta \\ \delta & 0 \end{pmatrix} \end{cases}, \quad (3)$$

where $w := \sqrt{v^2 + \delta^2}$ and $\lambda := \cot \sigma$. The Gaussian curvature K and the mean curvature H of Φ are given respectively by

$$K = \frac{-\delta^2}{w^4}, \quad H = -\frac{k v^2 + \delta'v + \delta^2(k + \lambda)}{2 w^3}. \quad (4)$$

Skew ruled surfaces Φ , whose osculating quadrics are rotational hyperboloids, are called *Edlinger-surfaces* [2], [3]. Necessary and sufficient conditions for a ruled surface Φ to be an Edlinger-surface are the following ([1]; p.103):

$$\delta' = k\lambda + 1 = 0.$$

This is a ruled surface of constant parameter of distribution whose striction line $s(u)$ is a line of curvature. The curves of *constant striction distance*, i.e. the curves $v = \text{constant}$, are in this case lines of curvature of Φ . The other family of the lines of curvature is determined by

$$[k^2 v^2 + \delta^2(k^2 + 1)] du - \delta k dv = 0.$$

It is easily verified that the corresponding normal curvatures of the lines of curvature (principal curvatures) are the following

$$k_1 = -k(u) w^{-1}, \quad k_2 = \frac{\delta^2(u)}{k(u)} w^{-3}. \quad (5)$$

In the rest of this paper *only skew (non-developable) ruled surfaces of the space \mathbb{R}^3 are considered with the parametrization (1) satisfying the conditions (2).*

2 The case of the principal curvatures

Starting point of this paragraph are the relations (5). Firstly the problem of finding all ruled surfaces whose a principal curvature has the following form is considered

$$k_i = f(u) w^n, \quad n \in \mathbb{Z}, \quad f(u) \in C^0(I), \quad i = 1 \text{ or } 2. \quad (6)$$

It is obvious that $f(u) \neq 0$ for all $u \in I$ since Φ is non-developable.

Using (3) the normal curvature in direction $du : dv$ is found to be

$$k_N = \frac{1}{w} \cdot \frac{-[kv^2 + \delta'v + \delta^2(k - \lambda)] du^2 + 2\delta du dv}{[v^2 + \delta^2(\lambda^2 + 1)] du^2 + 2\delta\lambda du dv + dv^2}. \quad (7)$$

Taking into account (6) it follows that

$$\begin{aligned} & [f w^{n+1} [v^2 + \delta^2(\lambda^2 + 1)] + k v^2 + \delta'v + \delta^2(k - \lambda)] du^2 \\ & + 2\delta(f\lambda w^{n+1} - 1) du dv + f w^{n+1} dv^2 = 0. \end{aligned}$$

This second order in $du : dv$ equation has *exactly one solution* if and only if its discriminant vanishes:

$$f^2 w^{2n+4} + f[kv^2 + \delta'v + \delta^2(k + \lambda)] w^{n+1} - \delta^2 = 0 \quad \forall u \in I, \quad v \in \mathbb{R}. \quad (8)$$

Let first be $n = 0$. Then (8) changes into

$$f^4(v^2 + \delta^2)^4 - 2f^2\delta^2(v^2 + \delta^2)^2 + \delta^4 - f^2[kv^2 + \delta'v + \delta^2(k + \lambda)]^2(v^2 + \delta^2) = 0.$$

In the left hand stays a polynomial of degree eight in v which vanishes for all $u \in I$ and infinite $v \in \mathbb{R}$. Comparing its coefficients with those of the zero polynomial it becomes obvious that f vanishes, which it was previously excluded.

We distinguish now the following cases:

Case I: Let $n \in \mathbb{Z}$ in (8) be odd. Then we have

$$\begin{aligned} Q(v) &:= f[kv^2 + \delta'v + \delta^2(k + \lambda)](v^2 + \delta^2)^{\frac{n+1}{2}} \\ &+ f^2(v^2 + \delta^2)^{n+2} - \delta^2 = 0 \quad \forall u \in I, \quad v \in \mathbb{R}. \end{aligned}$$

For $n \geq 1$ the vanishing of the coefficient of $v^{2(n+2)}$, which is the greatest power of v of the polynomial $Q(v)$, implies $f = 0$, which is impossible.

Let $n = -1$. Then

$$Q(v) = f^2(v^2 + \delta^2) + f[kv^2 + \delta'v + \delta^2(k + \lambda)] - \delta^2 = 0.$$

The vanishing of the coefficients of the polynomial $Q(v)$ gives

$$f = -k, \quad \delta' = k\lambda + 1 = 0,$$

therefore Φ is an Edlinger-surface.

Let $n = -3$ and k_1 the principal curvature having the form (6), i.e. $k_1 = f(u)w^{-3}$. Then from (4) for the other principal curvature the following expression is obtained

$$k_2 = f^*(u)w^{-1} \quad \text{with} \quad f^*(u) := \frac{-\delta^2(u)}{f(u)},$$

so that a principal curvature of Φ has the form (6), where $n = -1$. As it was previously established Φ is an Edlinger-surface.

The case $n \leq -5$ leads to a contradiction as one can easily confirm.

Case II: Let $n \in \mathbb{Z}$ in (8) be even. For $n = -2$ it follows from (8)

$$Q(v) := f^2[kv^2 + \delta'v + \delta^2(k + \lambda)]^2 - (f^2 - \delta^2)^2(v^2 + \delta^2) = 0 \quad \forall u \in I, v \in \mathbb{R}.$$

The vanishing of the coefficient f^2k^2 of v^4 implies $k = 0$. From the vanishing of the remaining coefficients of the polynomial $Q(v)$ the outcome is

$$f^2\delta'^2 - (f^2 - \delta^2)^2 = 2f^2\delta^2\delta'\lambda = f^2\delta^4\lambda^2 - \delta^2(f^2 - \delta^2)^2 = 0.$$

We finally obtain $\delta' = \lambda = 0$ and $f = \pm\delta$. Thus the surface Φ is a right helicoid.

The cases $n \geq 2$ and $n \leq 4$ lead to contradictions, as one can easily confirm.

The results are formulated as follows:

Proposition 1 *Let $\Phi \subset \mathbb{R}^3$ be a skew ruled C^3 -surface whose a principal curvature has the form (6). Then one of the following occurs:*

- (a) $n = -1$, $f(u) = -k(u)$ and Φ is an Edlinger-surface.
- (b) $n = -2$, $f(u) = \pm\delta(u)$ and Φ is a right helicoid.
- (c) $n = -3$, $f(u) = \delta^2(u)k^{-1}(u)$ and Φ is an Edlinger-surface.

From this proposition follows the next

Corollary 2 *Let $\Phi \subset \mathbb{R}^3$ be a skew ruled C^3 -surface, whose principal curvatures satisfy the relation*

$$\delta^2 k_1^3 + k^4 k_2 = 0. \tag{9}$$

Then Φ is an Edlinger-surface.

Proof. By using (4) and (9) we obtain $k_1^4 = k^4 w^{-4}$, so that it is $k_1 = \pm k w^{-1}$. From Proposition 1 it follows $k_1 = -k w^{-1}$. Thus the normal curvature k_1 has the required form (6), where $n = -1$. Hence Φ is an Edlinger-surface. ■

3 The case of the normal curvature

Continuing this line of work we consider further geometrically distinguished families of curves on the skew ruled surface Φ and suppose that the normal curvature along the curves of these families are of the form

$$k_N = f(u)w^n, \quad n \in \mathbb{Z}, \quad f(u) \in C^0(I). \tag{10}$$

Our aim is the specification of these ruled surfaces.

3.1. Let S_1 be the family of curves of constant striction distance. From (7) the normal curvature along a curve of S_1 is obtained

$$k_N = \frac{1}{w} \cdot \frac{-k v^2 - \delta' v + \delta^2 (k - \lambda)}{v^2 + \delta^2 (\lambda^2 + 1)}.$$

Therefore k_N has the form (10) if and only if

$$f w^{n+1} [v^2 + \delta^2 (\lambda^2 + 1)] + k v^2 + \delta' v + \delta^2 (k - \lambda) = 0. \quad (11)$$

It is observed that the function f vanishes exactly when for all $v \in \mathbb{R}$ holds

$$k v^2 + \delta' v + \delta^2 (k - \lambda) = 0, \quad (12)$$

so that $k = \delta' = \lambda = 0$, which means that Φ is a right helicoid.

Let now be $f \neq 0$. For $n = -1$, using (11) the following is derived

$$Q(v) := f[v^2 + \delta^2 (\lambda^2 + 1)] + k v^2 + \delta' v + \delta^2 (k - \lambda) = 0.$$

The vanishing of the coefficients of the polynomial $Q(v)$ implies

$$f = -k, \quad \delta' = 0, \quad \lambda (k\lambda + 1) = 0.$$

Therefore the surface Φ is either an orthoid ruled surface¹ of constant parameter of distribution ($\delta' = \lambda = 0$) or an Edlinger-surface ($\delta' = k\lambda + 1 = 0$).

For $n > -1$ it follows from (11)

$$Q(v) := f^2 (v^2 + \delta^2)^{n+1} [v^2 + \delta^2 (\lambda^2 + 1)]^2 - [k v^2 + \delta' v + \delta^2 (k - \lambda)]^2 = 0.$$

From the vanishing of the coefficient of $v^{2(n+3)}$ it follows that $f = 0$ which it was excluded.

For $n < -1$ it follows from (11)

$$Q(v) := (v^2 + \delta^2)^{-n-1} [k v^2 + \delta' v + \delta^2 (k - \lambda)]^2 - f^2 [v^2 + \delta^2 (\lambda^2 + 1)]^2 = 0.$$

The vanishing of the coefficient of $v^{2(n-1)}$ implies $k = 0$. Then the polynomial $Q(v)$ becomes

$$Q(v) = (v^2 + \delta^2)^{-n-1} (\delta' v - \delta^2 \lambda)^2 - f^2 [v^2 + \delta^2 (\lambda^2 + 1)]^2 = 0. \quad (13)$$

For $n = -2$ the polynomial $Q(v)$ takes the form

$$Q(v) = (v^2 + \delta^2) (\delta' v - \delta^2 \lambda)^2 - f^2 [v^2 + \delta^2 (\lambda^2 + 1)]^2 = 0.$$

From the vanishing of the coefficients of the polynomial $Q(v)$ the result is $f = 0$, which is a contradiction.

¹A ruled surface is called *orthoid* if its rulings are perpendicular to the striction line.

For $n < -2$ the vanishing of the coefficient of v^{-2n} in (13) implies $\delta' = 0$, therefore

$$Q(v) = \delta^4 \lambda^2 (v^2 + \delta^2)^{-n-1} - f^2 [v^2 + \delta^2 (\lambda^2 + 1)]^2 = 0. \quad (14)$$

In particular for $n = -3$ one has

$$Q(v) = \delta^4 \lambda^2 (v^2 + \delta^2)^2 - f^2 [v^2 + \delta^2 (\lambda^2 + 1)]^2 = 0.$$

From the vanishing of the coefficients of the polynomial $Q(v)$ it follows again that $f = 0$ which is a contradiction.

For $n < -3$ (14) results in $\lambda = f = 0$ which is equally impossible.

Thus the following has been shown:

Proposition 3 *Suppose that the normal curvature along the curves of constant striction distance of a skew ruled C^3 -surface $\Phi \subset \mathbb{R}^3$ has the form (10). Then one of the following occurs:*

- (a) $f = 0$ and Φ is a right helicoid.
- (b) $n = -1$, $f(u) = -k(u)$ and Φ is either an orthoid ruled surface of constant parameter of distribution or an Edlinger-surface.

3.2. Let S_2 be the family of the orthogonal trajectories of the family S_1 . This family is determined by

$$[v^2 + \delta^2 (\lambda^2 + 1)] du + \delta \lambda dv = 0.$$

From (7) the corresponding normal curvature is obtained

$$k_N = \frac{1}{w^3} \cdot \frac{-\delta^2 \lambda [(k \lambda + 2) v^2 + \delta' \lambda v + \delta^2 (\lambda^2 + k \lambda + 2)]}{v^2 + \delta^2 (\lambda^2 + 1)}.$$

Consequently k_N has the form (10) if and only if

$$f w^{n+3} [v^2 + \delta^2 (\lambda^2 + 1)] + \delta^2 \lambda [(k \lambda + 2) v^2 + \delta' \lambda v + \delta^2 (\lambda^2 + k \lambda + 2)] = 0. \quad (15)$$

Obviously the function f vanishes exactly when for all $v \in \mathbb{R}$ holds

$$\lambda [(k \lambda + 2) v^2 + \delta' \lambda v + \delta^2 (\lambda^2 + k \lambda + 2)] = 0.$$

In this case the function λ vanishes too, because otherwise it would have been

$$k \lambda + 2 = \delta' \lambda = \delta^2 (\lambda^2 + k \lambda + 2) = 0,$$

which are impossible. Therefore it is $f = 0$ if and only if $\lambda = 0$ and this means that Φ is an orthoid ruled surface.

Let now be $f \lambda \neq 0$. For $n = -3$ it follows from (15)

$$Q(v) := f [v^2 + \delta^2 (\lambda^2 + 1)] + \delta^2 \lambda [(k \lambda + 2) v^2 + \delta' \lambda v + \delta^2 (\lambda^2 + k \lambda + 2)] = 0.$$

From the vanishing of the coefficients of the polynomial $Q(v)$ it is obtained that

$$f = -\delta^2 \lambda (k \lambda + 2), \quad \delta' = 0, \quad k \lambda + 1 = 0.$$

This results in $f = -\delta^2 \lambda$ and Φ is an Edlinger-surface.

One can easily confirm that the cases $n > -3$ and $n < -3$ lead to contradictions. These results imply the following

Proposition 4 *Suppose that the normal curvature along the orthogonal trajectories of the curves of constant striction distance of a skew ruled C^3 -surface $\Phi \subset \mathbb{R}^3$ has the form (10). Then one of the following occurs:*

- (a) $f = 0$ and Φ is an orthoid ruled surface.
- (b) $n = -3$, $f(u) = \delta^2(u) k^{-1}(u)$ and Φ is an Edlinger-surface.

3.3. Let S_3 be the family of the orthogonal trajectories of the rulings, i.e. the family which is determined by

$$\delta \lambda du + dv = 0.$$

From (7) the corresponding normal curvature is obtained

$$k_N = \frac{-k v^2 - \delta' v - \delta^2 (k + \lambda)}{w^3}.$$

Therefore k_N has the form (10) if and only if

$$f w^{n+3} + k v^2 + \delta' v + \delta^2 (k + \lambda) = 0. \quad (16)$$

The function f vanishes if and only if (12) holds for all $v \in \mathbb{R}$ or, equivalently, if $k = \lambda = \delta' = 0$. Hence the surface Φ is a right helicoid.

Let now be $f \neq 0$. For $n = -3$ it follows from (16)

$$Q(v) := k v^2 + \delta' v + \delta^2 (k + \lambda) + f = 0.$$

The vanishing of the coefficients of the polynomial $Q(v)$ implies

$$f = -\delta^2 \lambda, \quad k = 0, \quad \delta' = 0.$$

Therefore Φ is a conoidal ruled surface of constant parameter of distribution.

For $n = -2$ it follows from (16)

$$Q(v) := f^2(v^2 + \delta^2) - [k v^2 + \delta' v + \delta^2 (k + \lambda)]^2 = 0.$$

From the vanishing of the coefficients of the polynomial $Q(v)$ it follows

$$k = 0, \quad f^2 = \delta'^2, \quad \delta' \lambda = 0, \quad f^2 = \delta^2 \lambda^2 = 0,$$

therefore $f = 0$ which it was excluded.

From (16) and for $n = -1$ the outcome is

$$Q(v) := f(v^2 + \delta^2) + k v^2 + \delta' v + \delta^2(k + \lambda) = 0.$$

From the vanishing of the coefficients of the polynomial $Q(v)$ it follows

$$f = -k, \quad \delta' = 0, \quad \lambda = 0.$$

Therefore Φ is an orthoid ruled surface of constant parameter of distribution.

One can easily confirm that the cases $n \geq 0$ and $n \leq -4$ lead to contradictions. So it can be stated that:

Proposition 5 *Suppose that the normal curvature along the orthogonal trajectories of the rulings of a skew ruled C^3 -surface $\Phi \subset \mathbb{R}^3$ has the form (10). Then one of the following occurs:*

- (a) $f = 0$ and Φ is a right helicoid.
- (b) $n = -3$, $f(u) = -\delta^2(u) \lambda(u)$ and Φ is a conoidal ruled surface of constant parameter of distribution.
- (c) $n = -1$, $f(u) = -k(u)$ and Φ is an orthoid ruled surface of constant parameter of distribution.

3.4. Let S_4 be the family of curves of constant Gaussian curvature [4]. This family is determined by

$$\delta'(\delta^2 - v^2) du + 2\delta v dv = 0.$$

Putting for abbreviation

$$A = (4\delta^2 + \delta'^2) v^4 + 4\delta^2 \delta' \lambda v^3 + 2\delta^2 [2\delta^2 (\lambda^2 + 1) - \delta'^2] v^2 - 4\delta^4 \delta' \lambda v + \delta^4 \delta'^2,$$

the corresponding normal curvature can be computed from (7) to be

$$k_N = \frac{-1}{w} \cdot \frac{4\delta^2 v [k v^3 + \delta^2 (k - \lambda) v + \delta^2 \delta']}{A}.$$

k_N has the form (10) if and only if

$$\begin{aligned} & f w^{n+1} [(4\delta^2 + \delta'^2) v^4 + 4\delta^2 \delta' \lambda v^3 + 2\delta^2 [2\delta^2 (\lambda^2 + 1) - \delta'^2] v^2 \\ & - 4\delta^4 \delta' \lambda v + \delta^4 \delta'^2] + 4\delta^2 v [k v^3 + \delta^2 (k - \lambda) v + \delta^2 \delta'] = 0. \end{aligned} \quad (17)$$

The function f vanishes exactly when

$$k v^3 + \delta^2 (k - \lambda) v + \delta^2 \delta' = 0$$

for all $v \in \mathbb{R}$ or, equivalently, if $k = \lambda = \delta' = 0$. Consequently Φ is a right helicoid.

Let now be $f \neq 0$. For $n = -1$ it follows from (17)

$$\begin{aligned} Q(v) := & f [(4\delta^2 + \delta'^2) v^4 + 4\delta^2 \delta' \lambda v^3 + 2\delta^2 [2\delta^2 (\lambda^2 + 1) - \delta'^2] v^2 \\ & - 4\delta^4 \delta' \lambda v + \delta^4 \delta'^2] + 4\delta^2 v [k v^3 + \delta^2 (k - \lambda) v + \delta^2 \delta'] = 0. \end{aligned}$$

The coefficients

$$\begin{aligned} a_4 &:= f(4\delta^2 + \delta'^2) + 4\delta^2 k, & a_3 &:= 4f\delta^2\delta'\lambda, \\ a_2 &:= 2f\delta^2[2\delta^2(\lambda^2 + 1) - \delta'^2] + 4\delta^4(k - \lambda), \\ a_1 &:= -4f\delta^4\delta'\lambda + 4\delta^4\delta', & a_0 &:= f\delta^4\delta'^2 \end{aligned}$$

of the polynomial $Q(v)$ vanish. From $a_0 = 0$ it follows $\delta' = 0$. Then from the vanishing of the coefficients a_2 and a_4 we obtain

$$f = -k, \quad \lambda(k\lambda + 1) = 0.$$

Consequently Φ is either an orthoid ruled surface of constant parameter of distribution ($\delta' = \lambda = 0$) or an Edlinger-surface ($\delta' = k\lambda + 1 = 0$).

The cases $n > -1$ and $n < -1$ lead to contradictions. The following has been shown

Proposition 6 *Suppose that the normal curvature along the curves of constant Gaussian curvature of a skew ruled C^3 -surface $\Phi \subset \mathbb{R}^3$ has the form (10). Then one of the following occurs:*

- (a) $f = 0$ and Φ is a right helicoid.
- (b) $n = -1$, $f(u) = -k(u)$ and Φ is either an orthoid ruled surface of constant parameter of distribution or an Edlinger-surface.

The following table assemble the above results.

Normal curvature of the form $k_N = f w^n$ along	f	n	Type of the ruled surface Φ
one family of the lines of curvature	$-k$ $\pm\delta$ $\delta^2 k^{-1}$	-1 -2 -3	\cdot Edlinger-surface \cdot right helicoid \cdot Edlinger-surface
the curves of const. striction distance	0 $-k$	$-$ -1	\cdot right helicoid \cdot either an orthoid surface of const. parameter of distrib. or an Edlinger-surface
the orthogonal trajectories of the curves of const. striction distance	0 $\delta^2 k^{-1}$	$-$ -3	\cdot orthoid surface \cdot Edlinger-surface
the orthogonal trajectories of the rulings	0 $-k$ $-\delta^2 \lambda$	$-$ -1 -3	\cdot right helicoid \cdot orthoid surface of const. parameter of distrib. \cdot conoidal surface of const. parameter of distrib.
the curves of const. Gaussian curvature	0 $-k$	$-$ -1	\cdot right helicoid \cdot either an orthoid surface of const. parameter of distrib. or an Edlinger-surface

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